Electrostatic in Reissner-Nordström space-time with a conical defect

J. Spinelly and V. B. Bezerra

Departamento de Física, Universidade Federal da Paraíba

Caixa Postal 5008, 58051-970 João Pessoa,PB,Brazil

We calculate the electrostatic potential generated by a point charge at rest in Reissner-Nordström space-time with a conical defect. An expression for the self-energy is also presented.

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It is well known that the gravitational field modifies the electrostatic interaction of a charged particle in such a way that the particle experiences a finite self-force [1,2]. The origin of this force comes from the space-time curvature associated with the gravitational field. On the other hand, even in the absence of curvature, such as in the conical space-time of an infinite straight cosmic string [3], it was shown that a charged point particle [4] or a linear charge distribution [5] placed at rest in this background becomes subject to a finite repulsive electrostatic self-force. In this case, the origin of this force is the distortion in the particle field caused by the lack of global flatness of the space-time of a cosmic string. Therefore, the modifications of the electrostatic potential comes from two contributions, one of geometric origin and the other of topological one.

Some authors have suggested that the most simple exact solutions of Einstein's equations can easily be generalized to include a conical defect [6]. Such space-times have been considered in different context and some investigations were done in these backgrounds [7].

In this paper we determine the expression for the electrostatic potential generated by a point charge held stationary in the space-time of Reissner- Nordström pierced by a cosmic string and also determine the self-energy. These results extend previous one obtained by Linet [8] in the case of a Schwarzschild background with conical defect.

The Reissner-Nordström space-time endowed with a conical defect takes the following

form

$$ds^2 = -\left(1 - \frac{2E}{br} + \frac{q^2}{b^2r^2}\right) dt^2 + \left(1 - \frac{2E}{br} + \frac{q^2}{b^2r^2}\right)^{-1} dr^2$$
$$+r^2 d\theta^2 + b^2r^2 \sin^2\theta d\varphi^2, \tag{1}$$

where b is a parameter quantifying the conical defect. Assuming that the conical defect is associated with a cosmic string, then the parameter b is given by $b = 1 - 4\mu$, where μ is the linear mass density of the cosmic string. The quantity E is the energy of the black hole [9] and q is the electric charge. Space-times with a conical defect are geometrically constructed by removing a wedge, that is, by requiring that the azimuthal angle around the axis runs over the range $0 < \phi < 2\pi b$. In the present case, in addition to the geometric procedure, we have to correct the mass term with the factor $\frac{1}{b}$ due to the fact that in the presence of a string, the energy at infinity and the Schwarzschild mass parameter are not identical [9]. The same correction must be applied to the electrostatic term due to Gauss theorem. In the calculations that follow we will consider the region of space-time outside the outer horizon, that is, for $r > r_+$, where $r_+ = \left(E + \sqrt{E^2 - q^2}\right)/b$, and assume that $E^2 > q^2$.

Consider a point test charge e, held stationary at $(r_o, \theta_o, \varphi_o)$, with $r_o > r_+$. In this case the current density J^i vanishes and the charge density J^0 is given by

$$J^0 = \frac{e}{br^2 \sin\theta} \delta(r - r_o)\, \delta(\theta - \theta_o)\, \delta(\varphi - \varphi_o)\,. \tag{2}$$

Now, let us take Maxwell's equations in the space-time given by metric (1) with source being a point test charge e. In the present case we have just one equation for the component A_0 which is given by

$$\partial_i \left(\sqrt{-g} F^{i0}\right) = 4\pi \sqrt{-g} J^0, \tag{3}$$

where $F_{i0} = \partial_i A_0$.

Therefore, in the space-time under consideration, the equation for A_0 can be written as

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_0}{\partial r}\right) + \left(1 - \frac{2E}{br} + \frac{q^2}{b^2r^2}\right)^{-1} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_0}{\partial\theta}\right) + \frac{1}{b^2\sin^2\theta}\frac{\partial^2 A_0}{\partial\varphi^2}\right]$$
$$= -\frac{4\pi e}{b\sin\theta} \delta(r - r_o)\, \delta(\theta - \theta_o)\, \delta(\varphi - \varphi_o)\,. \tag{4}$$

Doing the following substitutions [10]

$$r = r_s + r_- \tag{5}$$

$$A_0 = \frac{r - r_-}{r} A_0^s, \tag{6}$$

where $r_- = E/b - M$ and $M^2 = \left(E^2 - q^2\right)/b^2$, eq.(4) turns into

$$\begin{aligned} &\frac{\partial}{\partial r_s}\left(r_s^2 \frac{\partial A_0^s}{\partial r_s}\right) + \left(1 - \frac{2M}{r_s}\right)^{-1}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_0^s}{\partial\theta}\right) + \frac{1}{b^2\sin^2\theta}\frac{\partial^2 A_0^s}{\partial\varphi^2}\right] \\ &\qquad = -\frac{4\pi e}{b\sin\theta}\frac{r_s}{r_s + r_-}\delta\left(r_s - r_{so}\right)\delta\left(\theta - \theta_o\right)\delta\left(\varphi - \varphi_o\right), \end{aligned} \tag{7}$$

which can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial r_s}\left(r_s^2 \frac{\partial A_0^s}{\partial r_s}\right) + \left(1 - \frac{2M}{r_s}\right)^{-1}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_0^s}{\partial\theta}\right) + \frac{1}{b^2\sin^2\theta}\frac{\partial^2 A_0^s}{\partial\varphi^2}\right] \\ &\qquad = -\frac{4\pi e'}{b\sin\theta}\delta\left(r_s - r_{so}\right)\delta\left(\theta - \theta_o\right)\delta\left(\varphi - \varphi_o\right), \end{aligned} \tag{8}$$

where $e' = e r_{so}/\left(r_{so} + r_-\right)$. Equation (8) is formally identical to the partial differential equation for the electrostatic potential in the space-time of Schwarzschild with a conical defect [8]. Proceeding in analogy with Linet [8] and assuming that $\varphi_o = \pi$ and $1/2 < b < 1$, we find that the solution for (8) is given by

$$A_0^s\left(r_s, \theta, \varphi\right) = V_s^*\left(r_s, \theta, \varphi\right) + V_s^b\left(r_s, \theta, \varphi\right) + \frac{e' M}{b r_{so} r_s}, \tag{9}$$

where

$$V_s^*\left(r_s, \theta, \varphi\right) = \begin{cases} V_{cs}\left[r_s, \sigma_0\left(\theta, \varphi\right)\right] + V_{cs}\left[r_s, \sigma_1\left(\theta, \varphi\right)\right] & , \quad 0 < \varphi < \pi/b - \pi \\ V_{cs}\left[r_s, \sigma_0\left(\theta, \varphi\right)\right] & , \quad \pi/b - \pi < \varphi < 3\pi - \pi/b \\ V_{cs}\left[r_s, \sigma_0\left(\theta, \varphi\right)\right] + V_{cs}\left[r_s, \sigma_{-1}\left(\theta, \varphi\right)\right] & , \quad 3\pi - \pi/b < \varphi < 2\pi \end{cases}. \tag{10}$$

Note that solution given by eq.(9) is fixed taking into account the requirement that charge q of the black hole and charge e of the test particle contribute to the potential.

The term V_{cs} is the solution of Copson [11], which is a solution of eq.(8) for $b = 1$ and whose expression is [12]

$$V_{cs}\left[r_s,\sigma\right]=\frac{e'}{r_{so}r_s}\frac{\left(r_s-M\right)\left(r_{so}-M\right)-M^2\sigma}{\left[\left(r_s-M\right)^2+\left(r_{so}-M\right)^2-M^2-2\left(r_s-M\right)\left(r_{so}-M\right)\sigma+M^2\sigma^2\right]^{1/2}} \quad (11)$$

with

$$\sigma_0\left(\theta,\varphi\right)=\cos\theta\cos\theta_o+\sin\theta\sin\theta_o\cos[b\left(\varphi-\pi\right)] \quad (12)$$

$$\sigma_1\left(\theta,\varphi\right)=\cos\theta\cos\theta_o+\sin\theta\sin\theta_o\cos[b\left(\varphi+\pi\right)] \quad (13)$$

$$\sigma_{-1}\left(\theta,\varphi\right)=\cos\theta\cos\theta_o+\sin\theta\sin\theta_o\cos[b\left(\varphi-3\pi\right)]. \quad (14)$$

The potential V_s^b that appears in eq.(9) is given by [8]

$$V_s^b\left(r_s,\theta,\varphi\right)=\frac{1}{2\pi b}\int_0^\infty V_{cs}\left[r_s,k\left(\theta,x\right)\right]F_b\left(\varphi,x\right)dx, \quad (15)$$

where $k\left(\theta,x\right)$ and $F_b\left(\varphi,x\right)$ are

$$k\left(\theta,x\right)=\cos\theta\cos\theta_o-\sin\theta\sin\theta_o\cosh x \quad (16)$$

and

$$F_b\left(\varphi,x\right)=-\frac{\sin\left(\varphi-\pi/b\right)}{\cosh\left(x/b\right)+\cos\left(\varphi-\pi/b\right)}+\frac{\sin\left(\varphi+\pi/b\right)}{\cosh\left(x/b\right)+\cos\left(\varphi+\pi/b\right)}. \quad (17)$$

Now, using eqs. (5) and (6), we can rewrite the electrostatic potential given by eq.(9) as

$$A_0=V^*\left(r,\theta,\varphi\right)+V^b\left(r,\theta,\varphi\right)+\frac{eM}{br_or}, \quad (18)$$

where

$$V^*\left(r,\theta,\varphi\right)=\begin{cases} V_c\left[r,\sigma_0\left(\theta,\varphi\right)\right]+V_c\left[r,\sigma_1\left(\theta,\varphi\right)\right] & , \quad 0<\varphi<\pi/b-\pi \\ V_c\left[r,\sigma_0\left(\theta,\varphi\right)\right] & , \quad \pi/b-\pi<\varphi<3\pi-\pi/b\ , \\ V_c\left[r,\sigma_0\left(\theta,\varphi\right)\right]+V_c\left[r,\sigma_{-1}\left(\theta,\varphi\right)\right] & , \quad 3\pi-\pi/b<\varphi<2\pi \end{cases} \quad (19)$$

and V_c

$$V_c\left[r,\sigma\right]=\frac{e}{r_or}\frac{\left(r-E/b\right)\left(r_o-E/b\right)-M^2\sigma}{\left[\left(r-E/b\right)^2+\left(r_o-E/b\right)^2-M^2-2\left(r-E/b\right)\left(r_o-E/b\right)\sigma+M^2\sigma^2\right]^{1/2}}. \quad (20)$$

The potential V^b obeys the following relation

$$V^b(r,\theta,\varphi) = \frac{1}{2\pi b}\int_0^\infty V_c\left[r, k(\theta,x)\right] F_b(\varphi,x)\,dx. \tag{21}$$

Doing the same procedure as in [8] we can verify that eq.(9) is a solution of eq.(8), or returning to the original radial coordinate, we conclude that eq.(18) is a solution of eq.(4), that is, eq.(18) is the electrostatic potential generated by a point charge e placed at rest in the space-time of a charged black hole (Reissner-Nordström space-time) with a cosmic string passing through it.

Now, let us consider the electrostatic potential eq.(18) in a neighborhood of the point (r_o, θ_o, π) in the region defined by $\frac{\pi}{b} < \phi < 3\pi - \frac{\pi}{b}$. The terms $V^b(r,\theta)$ and eM/brr_o are finite at the position of the test charge. The term $V^*(r,\theta)$ is infinite at that point and corresponds to the Coulombian potential. Therefore the electrostatic self-force comes from $V^{reg} \equiv V^b(r,\theta) + eM/brr_o$. From the definition of energy for an arbitrary charge distribution we get that the self-energy is given by

$$U_{self}(r_o,\theta_o) = \frac{1}{2}\left(eV^b(r_o,\theta_o) + \frac{e^2 M}{br_o^2}\right). \tag{22}$$

Putting eq. (21) into eq.(22) and using eq.(17), we get the following result

$$U_{self}(r_o,\theta_o) = \frac{e^2 M}{2br_o^2} - \frac{e\sin(\pi/b)}{2\pi}\int_0^\infty V_c\left[r_o, k(\theta_o,x)\right]\frac{dx}{\cosh(x/b) - \cos(\pi/b)} \tag{23}$$

Equation (23) is the gravitationally induced electrostatic self-energy on a test point particle at rest in the space-time of Reissner-Nordström with a conical defect(cosmic string). It is important to call attention to the fact that this result brings together the geometric and topological aspects. The geometric contribution is connected with the curvature of the space-time and the topological features are due to the lack of spherical symmetry produced by the conical defect.

From the present results we can get the particular cases which has already been obtained corresponding to the electrostatic self-force in the Schwarzschild black hole with [8] and without [2] a conical defect and in the cosmic string space-time [4].

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